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ON POSITIVE CONVOLUTION OPERATORS FOR
JACOBI SERIES

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On positive convolution operators for Jacobi series

by

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1. Introduction

1.1. In a preceding paper [2] the author has started the study of approximation of functions by processes, which are generated by the use of summability methods for the expansion of the functions in terms of Jacobi polynomials. The summability methods can be interpreted as convolution operators, if the convolution structure for Jacobi series, defined by Askey and Wainger [1], is used. By means of some general theorems on approximation processes in Banach spaces, (Berens [3]), it is possible to characterize the saturation class and the classes of non-optimal approximation of a number of classical summability methods for the summation of the Fourier-Jacobi series. This paper deals with saturation of positive convolution operators and the main part is a theorem of the Tureckii [10] - DeVore [4] type, which determines the saturation order and the saturation class of a sequence of positive convolution operators, satisfying a special condition on the Fourier-Jacobi coefficients of the kernel. The proof is a straightforward generalization of DeVore's proof in the case of Fourier series. As applications, the saturation class of the higher order Jackson kernel and some other positive kernels are characterized.

1.2. We introduce same Banach spaces of complex valued functions on the interval [-1,1]. We write C for the space of continuous functions, L^{∞} denotes the space of essentially bounded functions and we define the L^p spaces with respect to the weight function ($x = \cos\theta$)

$$(1.1) \qquad \rho^{(\alpha,\beta)}(\theta) = (\sin\tfrac{\theta}{2})^{2\alpha+1}\,(\cos\tfrac{\theta}{2})^{2\beta+1} \qquad (\alpha \geq \beta \geq -\tfrac{1}{2}).$$

We call M the space of all regular finite Borel measures on [-1,1]. The spaces C, L^p ($1 \le p \le \infty$) and M are Banach spaces if endowed with the following norms

$$||f||_C = \sup_{0\le\theta\le\pi} |f(\cos\theta)|,$$

$$||f||_p = [\int_0^\pi |f(\cos\theta)|^p \rho^{(\alpha,\beta)}(\theta)\, d\theta]^{1/p} \qquad (1 \le p < \infty),$$

$$||f||_\infty = \operatorname{ess\,sup}_{0\le\theta\le\pi} |f(\cos\theta)|,$$

$$||\mu||_M = \int_0^\pi |d\mu(\cos\theta)|.$$

With elements of these Banach spaces we can associate an expansion in terms of Jacobi polynomials. If $P_n^{(\alpha,\beta)}(x)$ is written for the Jacobi polynomial of degree n and order (α,β) (see Szegö [9]), the functions

$$R_n^{(\alpha,\beta)}(\cos\theta) = \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)}$$

satisfy

$$(1.2) \qquad \int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\, R_m^{(\alpha,\beta)}(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)\, d\theta = \delta_{n,m}[\omega_n^{(\alpha,\beta)}]^{-1}.$$

Here,

$$(1.3) \qquad \omega_n^{(\alpha,\beta)} = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\Gamma(n+1)\Gamma(\alpha+1)\Gamma(\alpha+1)} = O(n^{2\alpha+1}) \qquad (n\to\infty).$$

With f belonging to one of the spaces C or L^p ($1 \le p \le \infty$) we associate the Fourier-Jacobi expansion

$$(1.4) \qquad f(\cos\theta) \sim \sum_{n=0}^{\infty} f^{\wedge}(n)\, \omega_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta),$$

where

$$(1.5)\qquad f^{\wedge}(n) = \int_0^{\pi} f(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)\, d\theta \qquad (n=0,1,\ldots).$$

With a measure $\mu \in M$ we associate the Jacobi-Stieltjes expansion

$$(1.6)\qquad d\mu(\cos\theta) \sim \sum_{n=0}^{\infty} \mu^{\vee}(n)\, \omega_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta),$$

where

$$(1.7)\qquad \mu^{\vee}(n) = \int_0^{\pi} R_n^{(\alpha,\beta)}(\cos\theta)\, d\mu(\cos\theta) \qquad (n=0,1,\ldots).$$

Askey and Wainger [1] have introduced a generalized translation operator T_ϕ, which maps a function f with (1.4) into

$$(1.8)\qquad T_\phi\, f(\cos\theta) \sim \sum_{n=0}^{\infty} f^{\wedge}(n)\, \omega_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\phi),$$

and Gasper [5] has shown the positivity of this operator. This implies that T_ϕ has an operator norm 1. If $f_1, f_2 \in L^1$, then the convolution $f_1 * f_2$ is defined by

$$(1.9)\qquad (f_1 * f_2)(\cos\theta) = \int_0^{\pi} T_\phi\, f_1(\cos\theta)\, f_2(\cos\phi)\, \rho^{(\alpha,\beta)}(\phi)\, d\phi.$$

This convolution has the usual properties (see Gasper [5]). If $f \in L^p$ $(1 \leq p \leq \infty)$ and $\mu \in M$ we can define the convolution $f * d\mu$ by

$$(1.10)\qquad (f * d\mu)(\cos\theta) = \int_0^{\pi} T_\phi\, f(\cos\theta)\, d\mu(\cos\phi).$$

Moreover, $f * d\mu \in L^p$ and the following inequality holds

$$(1.11)\qquad ||f * d\mu||_p \leq ||f||_p\, ||\mu||_M.$$

1.3. In the rest of this paper X is written for one of the spaces C or L^p $(1 \leq p < \infty)$. Assume that we are given a sequence $\{L_n\}$ of positive convolution operators, that is, L_n has the form

$$(1.12)\quad L_n(f;\cos\theta) = (f*d\mu_n)(\cos\theta) = \int_0^\pi T_\phi\, f(\cos\theta)\, d\mu_n(\cos\phi) \qquad (f\in X),$$

where μ_n $(n=1,2,\ldots)$ are non-negative elements of M with $\int_0^\pi d\mu_n(\cos\phi) = 1$.

We say that the sequence $\{L_n\}$ is saturated if there exists a non-increasing sequence of positive numbers $\{\phi(n)\}$ with $\lim_{n\to\infty} \phi(n) = 0$, such that

i) $||f-L_n(f)||_X = o(\phi(n))$ $(n\to\infty)$

if and only if f belongs to some "trivial" subspace of X

and

ii) there is a "non-trivial" element $f_0 \in X$ satisfying

$||f_0-L_n(f_0)||_X = O(\phi(n))$ $(n\to\infty)$.

The sequence $\{\phi(n)\}$ is then called the saturation order and the set $F(X,L_n)$, which consists of all the elements of X which satisfy ii, is called the saturation class or Favard class of L_n.

In this paper we shall prove a theorem, in which the behavior of the second trigonometric moment

$$(1.13)\qquad T(\mu_n;2) = \int_0^\pi \left(\sin\frac{\theta}{2}\right)^2 d\mu_n(\cos\theta)$$

determines the saturation of $\{L_n\}$. In section 2 we give some inequalities for Jacobi polynomials and we investigate the relationship between Jacobi coefficients and trigonometric moments. Then, following DeVore [4], we introduce the following conditions:

A. There exists a constant $C_A > 0$ such that for each integer k there is an $N(k)$ for which

$$\frac{1-\check{\mu}_n(k)}{1-\check{\mu}_n(1)} \geq C_A\, k(k+\alpha+\beta+1) \qquad \text{for } n > N(k).$$

B. There exists a constant $C_B > 0$ such that for each $\varepsilon > 0$ there is an $N(\varepsilon)$ such that

$$\int_0^{\varepsilon} (\sin \tfrac{\theta}{2})^2 \, d\mu_n(\cos\theta) \geq C_B \int_0^{\pi} (\sin \tfrac{\theta}{2})^2 \, d\mu_n(\cos\theta) \qquad \text{for } n > N(\varepsilon).$$

In section 3 we shall prove

1.4. Lemma. The conditions A and B are equivalent.

We define the Lipschitz classes with respect to the generalized translation operator by

$$(1.14)\quad \mathrm{Lip}(\gamma,X) = \{f\in X:\ \exists\, c > 0,\ \sup_{0<\psi<\phi} \|T_\psi f - f\|_X \leq c\phi^\gamma\},\quad (0 < \gamma \leq 2).$$

We now state the following theorem that will be proved in section 4.

1.5. Theorem. If $\{L_n\}$ is a sequence of operators of the form (1.12) and if either condition A or condition B is satisfied, then $\{L_n\}$ is saturated with order $(1-\check{\mu}_n(1))$ and the saturation class $F(X,L_n)$ is Lip(2,X).

The Jacobi polynomials $R_n^{(\alpha,\beta)}(\cos\theta)$ satisfy the following differential equation:

$$(1.15)\quad -\frac{1}{\rho^{(\alpha,\beta)}(\theta)} \frac{d}{d\theta} \{\rho^{(\alpha,\beta)}(\theta) \frac{d}{d\theta} R_n^{(\alpha,\beta)}(\cos\theta)\} = n(n+\alpha+\beta+1)\, R_n^{(\alpha,\beta)}(\cos\theta).$$

If for $f \in X$ with the expansion (1.4) there exists an element $Af \in X$ such that

$$(1.16)\quad Af \sim \sum_{n=0}^{\infty} n(n+\alpha+\beta+1)\, f^\wedge(n)\, \omega_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta),$$

then we say that $f \in D(A)$ and we call A the operator which maps D(A) into X by $f \to Af$. The operator is the realization in X of the differential operator

$$-\frac{1}{\rho^{(\alpha,\beta)}(\theta)}\frac{d}{d\theta}\{\rho^{(\alpha,\beta)}(\theta)\frac{d}{d\theta}\}$$

with boundary contitions $\frac{d}{d\theta} = 0$ at $\theta = 0$ and π, as follows from (1.15). Löfström and Peetre [7] have shown the close connection between the generalized translation operator T_ϕ and the operator A. In fact, for $f \in D(A)$ the following relations hold:

$$(1.17) \qquad ||T_\phi f-f||_X \leq C_1(\phi)\ ||Af||_X,$$

$$(1.18) \qquad \lim_{\phi\to 0^+} ||\frac{f - T_\phi f}{C_1(\phi)} - Af||_X = 0,$$

where

$$(1.19) \qquad C_1(\phi) = \int_0^\phi \frac{1}{\rho^{(\alpha,\beta)}(\theta)} (\int_0^\theta \rho^{(\alpha,\beta)}(\tau)\ d\tau)\ d\theta,$$

(see Bavinck [2], section 4). Moreover,

$$(1.20) \qquad \lim_{\phi\to 0^+} \frac{C_1(\phi)}{\sin^2\frac{\phi}{2}} = \frac{1}{\alpha + 1}$$

and, since for $0 < \phi \leq \frac{\pi}{2}$, $\frac{\sqrt{2}}{2} \leq \cos\frac{\phi}{2} \leq 1$ we have

$$C_1(\phi) \leq \int_0^\phi \frac{1}{\rho^{(\alpha,\beta)}(\theta)} \int_0^\theta (\sin\frac{\tau}{2})^{2\alpha+1} \cos\frac{\tau}{2}\ d\tau\ d\theta$$

$$(1.21) \qquad \leq \frac{1}{\alpha+1}\int_0^\phi \frac{\sin\frac{\theta}{2}}{(\cos\frac{\theta}{2})^{2\beta+1}}\ d\theta$$

$$\leq \frac{2^{\beta+1}}{\alpha+1}\sin^2\frac{\phi}{2} \qquad 0 < \phi \leq \frac{\pi}{2}.$$

Notation: We will use the notation $a_n \approx b_n$ $(n\to\infty)$ if there are positive numbers c_1 and c_2 such that $c_1 a_n \leq b_n \leq c_2 a_n$.

2. Some relations for Jacobi polynomials

2.1. Inequalities

We shall first prove the following inequaltities for Jacobi polynomials $R_k^{(\alpha,\beta)}(x)$. Let k be a natural number. Then

$$(2.1) \qquad 1 - R_k^{(\alpha,\beta)}(\cos\theta) \le \frac{k(k+\alpha+\beta+1)}{\alpha+1}\sin^2\frac{\theta}{2} \qquad (0 \le \theta \le \pi).$$

There exists a constant $c_\alpha > 0$, such that for $0 < \varepsilon < 4/2k+\alpha+\beta+2$

$$(2.2) \qquad c_\alpha \frac{k(k+\alpha+\beta+1)}{\alpha+1}\sin^2\frac{\theta}{2} \le 1 - R_k^{(\alpha,\beta)}(\cos\theta) \qquad (0 \le \theta \le \varepsilon).$$

By the differentiation formula

$$\frac{d}{dx} R_k^{(\alpha,\beta)}(x) = \frac{k(k+\alpha+\beta+1)}{2(\alpha+1)} R_{k-1}^{(\alpha+1,\beta+1)}(x)$$

we obtain from the mean-value theorem

$$(2.3) \qquad 1 - R_k^{(\alpha,\beta)}(\cos\theta) = \frac{k(k+\alpha+\beta+1)}{(\alpha+1)}\sin^2\frac{\theta}{2} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\bar{\theta}),\ 0 \le \bar{\theta} \le \theta.$$

Since $|R_{k-1}^{(\alpha+1,\beta+1)}(\cos\bar{\theta})| \le 1$, $0 \le \bar{\theta} \le \pi$, formula (2.1) follows.

For the proof of (2.2) we use Hilb's formula (Szegö [9], (8,21.12) for large n

$$(\sin\frac{\theta}{2})^\alpha (\cos\frac{\theta}{2})^\beta R_n^{(\alpha,\beta)}(\cos\theta) = N^{-\alpha}\,\Gamma(\alpha+1)(\theta/\sin\theta)^{\frac{1}{2}} J_\alpha(N\theta)$$
$$+ \begin{cases} \theta^{\frac{1}{2}}\, O(n^{-3/2-\alpha}), & \text{if } cn^{-1} \le \theta \le \pi-\varepsilon, \\ \theta^{\alpha+2}\, O(1), & \text{if } 0 < \theta \le cn^{-1}, \end{cases}$$

where $N = n + (\alpha+\beta+1)/2$.

The power series expansion of $(\frac{z}{2})^{-\alpha} J_\alpha(z)$ has terms with alternating sign, and monotonically decreasing for real z, $0 < z < 2$. Hence we have

$$R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta) \geq \Gamma(\alpha+2)\left(\frac{2}{N\theta}\right)^{\alpha+1} J_{\alpha+1}(N\theta) + \theta^2\, O(1) \qquad 0 \leq \theta < 2N^{-1}$$

$$(2.4) \qquad \geq 1 - \frac{\left(\frac{N\theta}{2}\right)^2}{\alpha+2} + \theta^2\, O(1)$$

$$> \frac{\alpha+1}{\alpha+2} - O(N^{-2}).$$

The inequality (2.2) follows from (2.3) and (2.4) for $k \geq k_0$. On the other hand, the constant c_α can be chosen in such a way, that (2.2) remains valid for $k \leq k_0$.

2.2. Relations between trigonometric moments and Jacobi coefficients

The following expansion is a simple consequence of Rodrigues' formula (see also Szegö [9], formula (9.3.11)).

$$(2.5) \quad \left(\sin\frac{\theta}{2}\right)^{2\sigma} =$$

$$= \frac{\Gamma(\sigma+1)\Gamma(\sigma+\alpha+1)}{\Gamma(\alpha+1)} \sum_{n=0}^{\sigma} (-1)^n \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\sigma-n+1)\Gamma(n+\alpha+\beta+\sigma+2)\Gamma(n+1)} R_n^{(\alpha,\beta)}(\cos\theta)$$

$$(\sigma=1,2,\ldots).$$

From the expression of the Jacobi polynomials in terms of hypergeometric functions

$$R_n^{(\alpha,\beta)}(\cos\theta) = {}_2F_1(-n,n+\alpha+\beta+1);\alpha+1;\sin^2\frac{\theta}{2})$$

we easily derive

$$(2.6) \quad 1 - R_n^{(\alpha,\beta)}(\cos\theta) =$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \frac{\Gamma(n+\alpha+\beta+k+1)\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n-k+1)\Gamma(n+\alpha+\beta+1)\Gamma(k+\alpha+1)\Gamma(k+1)} \sin^{2k}\frac{\theta}{2}\,.$$

If the trigonometric moment of order 2σ ($\sigma=1,2,\ldots$) is defined by

$$T(\mu_n;2\sigma) = \int_0^\pi (\sin \frac{\theta}{2})^2 \, d\mu_n(\cos \theta),$$

we obtain by (2.5), noticing the value of (2.5) at $\theta = 0$,

(2.7) $T(\mu_n;2\sigma) =$

$$= \frac{\Gamma(\sigma+1)\Gamma(\sigma+\alpha+1)}{\Gamma(\alpha+1)} \sum_{k=1}^{\sigma} (-1)^{k+1} \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(\sigma-k+1)\Gamma(k+\alpha+\beta+\sigma+2)\Gamma(k+1)} (1-\check{\mu}_n(k)).$$

On the other hand (2.6) leads to

(2.8) $1 - \check{\mu}_n(k) =$

$$= \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+\beta+1)} \sum_{\sigma=1}^{k} (-1)^{\sigma+1} \frac{\Gamma(k+\alpha+\beta+\sigma+1)}{\Gamma(k-\sigma+1)\Gamma(\sigma+\alpha+1)\Gamma(\sigma+1)} T(\mu_n;2\sigma).$$

Hence, we easily derive from (2.7)

$$(2.9) \qquad T(\mu_n;2) = \frac{\alpha + 1}{\alpha + \beta + 2} (1-\check{\mu}_n(1))$$

and

$$(2.10) \quad \frac{T(\mu_n;4)}{T(\mu_n;2)} = \frac{(\alpha+2)(\alpha+\beta+2)}{(\alpha+\beta+3)(\alpha+\beta+4)} \left[\frac{2(\alpha+\beta+3)}{\alpha+\beta+2} - \frac{1-\check{\mu}_n(2)}{1-\check{\mu}_n(1)} \right].$$

From (2.8) and (2.9) we conclude

(2.11) $\dfrac{1 - \check{\mu}_n(k)}{1 - \check{\mu}_n(1)} =$

$$\frac{k(k+\alpha+\beta+1)}{\alpha+\beta+2} - \frac{\Gamma(k+1)\Gamma(\alpha+2)}{(\alpha+\beta+2)\Gamma(k+\alpha+\beta+1)} \sum_{\sigma=2}^{k} (-1)^{\sigma} \frac{\Gamma(k+\alpha+\beta+\sigma+1)}{\Gamma(k-\sigma+1)\Gamma(\sigma+\alpha+1)\Gamma(\sigma+1)} \frac{T(\mu_n;2\sigma)}{T(\mu_n;2)}.$$

Similar relations between trigonometric moments and Fourier coefficients have been established by Stark [8]. We also have the following theorem, which generalizes a result of Görlich and Stark [6] (see also Stark [8]).

2.3. Theorem. For a sequence $\{L_n\}$ of positive convolution operators of the form (1.12) the following assertions are equivalent:

(a) $$\lim_{n\to\infty} \frac{1-\check{\mu}_n(k)}{1-\check{\mu}_n(1)} = \frac{k(k+\alpha+\beta+1)}{\alpha+\beta+2} \qquad (k=1,2,\ldots),$$

(b) $$\lim_{n\to\infty} \frac{1-\check{\mu}_n(2)}{1-\check{\mu}_n(1)} = \frac{2(\alpha+\beta+3)}{\alpha+\beta+2},$$

(c) $$\lim_{n\to\infty} \frac{T(\mu_n;4)}{T(\mu_n;2)} = 0.$$

Proof. Relation (b) is a trivial consequence of (a). Relation (c) follows from (b) by (2.10). Since $0 \le \sin^2 \frac{\theta}{2} \le 1$ and the measures μ_n are positive it is obvious that

$$T(\mu_n;2\sigma) \le T(\mu_n;4) \qquad \text{for } \sigma \ge 2.$$

Therefore relation (c) implies that $\lim_{n\to\infty} \frac{T(\mu_n;2\sigma)}{T(\mu_n;2)} = 0$, $\sigma \ge 2$. Thus, by formula (2.11) relation (a) follows.

3. Proof of lemma 1.4.

We first show that B implies A. If we take $\varepsilon < \frac{4}{2k+\alpha+\beta+2}$ and $N(\varepsilon)$ as given in B, we have using (2.2) and (2.9) for $n > N(\varepsilon)$

$$1 - \overset{\vee}{\mu}_n(k) = \int_0^\pi (1 - R_k^{(\alpha,\beta)}(\cos\theta)\, d\mu_n(\cos\theta)$$

$$\geq \int_0^\varepsilon (1 - R_k^{(\alpha,\beta)}(\cos\theta))\, d\mu_n(\cos\theta)$$

$$\geq c_\alpha \frac{k(k+\alpha+\beta+1)}{\alpha+1} \int_0^\varepsilon \sin^2\frac{\theta}{2}\, d\mu_n(\cos\theta)$$

$$\geq c_\alpha \frac{k(k+\alpha+\beta+1)}{\alpha+1} C_B \int_0^\pi \sin^2\frac{\theta}{2}\, d\mu_n(\cos\theta)$$

$$= \frac{c_\alpha C_B}{\alpha+\beta+2} k(k+\alpha+\beta+1)\,(1 - \overset{\vee}{\mu}_n(1)).$$

Therefore, A holds with $N(k) = N(\varepsilon)$ and $C_A = \frac{c_\alpha C_B}{\alpha+\beta+2}$.

We will now show that A implies B with $C_B = C_A \frac{(\alpha+\beta+2)}{2}$. Suppose B does not hold for $C_B = C_A \frac{(\alpha+\beta+2)}{2}$, then there is an $\varepsilon_0 > 0$ and a sequence (n_j) such that

$$(3.1)\quad \int_0^{\varepsilon_0} \sin^2\frac{\theta}{2}\, d\mu_{n_j}(\cos\theta) < C_A \frac{(\alpha+\beta+2)}{2} \int_0^\pi \sin^2\frac{\theta}{2}\, d\mu_{n_j}(\cos\theta), \quad j=1,2,\ldots .$$

We consider the measures

$$\nu_{n_j}(\cos\theta) = \begin{cases} 0, & 0 \leq \theta < \varepsilon_0, \\ \frac{1}{T(\mu_{n_j};2)}\mu_{n_j}(\cos\theta), & \varepsilon_0 \leq \theta \leq \pi. \end{cases}$$

Then $\int_0^\pi d\nu_{n_j}(\cos\theta) \le \frac{1}{\sin^2\frac{\varepsilon_0}{2}} \frac{1}{T(\mu_{n_j};2)} \int_0^\pi \sin^2\frac{\theta}{2}\, d\mu_{n_j}(\cos\theta) = \frac{1}{\sin^2\frac{\varepsilon_0}{2}}$.

By the weak* compactness of a closed sphere in M there exists a subsequence $(n_j') \subseteq (n_j)$ and a measure ν such that $\nu_{n_j'}$ converges weak* to ν. In particular we have for each k (k=1,2,...)

$$\lim_{n_j'\to\infty} \int_0^\pi \{1-R_k^{(\alpha,\beta)}(\cos\theta)\}\, d\nu_{n_j'}(\cos\theta) = \int_0^\pi \{1-R_k^{(\alpha,\beta)}(\cos\theta)\}\, d\nu \le 2\int_0^\pi d\nu.$$

Choose k_0 so large that

$$\frac{C_A\, k_0(k_0+\alpha+\beta+1)(\alpha+\beta+2)}{4(\alpha+1)} \ge \int_0^\pi d\nu. \tag{3.2}$$

Then there exists an N such that for $n_j' \ge N$

$$\frac{1}{T(\mu_{n_j'};2)} \int_0^{\varepsilon_0} \{1-R_{k_0}^{(\alpha,\beta)}(\cos\theta)\}\, d\mu_{n_j'}(\cos\theta) =$$

$$= \frac{1}{T(\mu_{n_j'};2)} \int_0^\pi \{1-R_{k_0}^{(\alpha,\beta)}(\cos\theta)\}\, d\mu_{n_j'}(\cos\theta)$$

$$- \int_0^\pi \{1-R_{k_0}^{(\alpha,\beta)}(\cos\theta)\}\, d\nu_{n_j'}(\cos\theta)$$

$$\ge \frac{1}{T(\mu_{n_j'};2)} \int_0^\pi \{1-R_{k_0}^{(\alpha,\beta)}(\cos\theta)\}\, d\mu_{n_j'}(\cos\theta)$$

$$- \frac{C_A\, k_0(k_0+\alpha+\beta+1)(\alpha+\beta+2)}{2(\alpha+1)}.$$

By virtue of condition A we have for $n_j' \ge \max(N, N(k_0))$

$$\int_0^{\varepsilon_0} \{1-R_{k_0}^{(\alpha,\beta)}(\cos\theta)\}\, d\mu_{n_j'}(\cos\theta) \geq$$

$$\geq C_A\, k_0(k_0+\alpha+\beta+1)\, \frac{(\alpha+\beta+2)}{2(\alpha+1)} \int_0^{\pi} \sin^2\frac{\theta}{2}\, d\mu_{n_j'}(\cos\theta).$$

Finally, by (2.1) we have

$$\int_0^{\varepsilon_0} \sin^2\frac{\theta}{2}\, d\mu_{n_j'}(\cos\theta) \geq \frac{(\alpha+1)}{k_0(k_0+\alpha+\beta+1)} \int_0^{\varepsilon_0} \{1-R_{k_0}^{(\alpha,\beta)}(\cos\theta)\}\, d\mu_{n_j'}(\cos\theta)$$

$$\geq C_A\, \frac{(\alpha+\beta+2)}{2} \int_0^{\pi} \sin^2\frac{\theta}{2}\, d\mu_{n_j'}(\cos\theta),$$

which is a contradiction to (3.1) and proves lemma 1.4.

4. Proof of theorem 1.5

Let $\{L_n\}$ be a sequence of positive linear operators of the form (1.12) which satisfy either condition A or B. On account of lemma 1.4 both conditions A and B are satisfied and we will interchange them appropriately.

We first show that $\{L_n\}$ is saturated with order $(1-\overset{\vee}{\mu}_n(1))$. If $f \in X$ and

$$||L_n(f)-f||_X = o(1-\overset{\vee}{\mu}_n(1)) \qquad (n\to\infty),$$

then

$$f^{\wedge}(k) - f^{\wedge}(k)\overset{\vee}{\mu}_n(k) = o(1-\overset{\vee}{\mu}_n(1)) \qquad (n\to\infty).$$

In view of condition A this implies $f^{\wedge}(k) = 0$, $k = 1,2,\ldots$, and therefore f is a constant. The function $f_0(\cos\theta) = (\sin\frac{\theta}{2})^2$ is an example of a non-constant function which satisfies

$$||L_n(f)-f||_X = O(1-\overset{\vee}{\mu}_n(1)) \qquad (n\to\infty).$$

Hence $\{L_n\}$ is saturated with order $(1-\overset{\vee}{\mu}_n(1))$. The "trivial" subspace used in section 1.3 is here the space of constant functions.

We now whish to characterize the saturation class $F(X,L_n)$. An element $f \in X$ belongs to $F(X,L_n)$ if and only if

$$||\int_0^{\pi} (T_{\phi}f(\cos\theta)-f(\cos\theta))\, d\mu_n(\cos\phi)||_X = O(1-\overset{\vee}{\mu}_n(1)) \quad (n\to\infty),$$

or equivalently

$$||\int_0^{\pi} \frac{(T_{\phi}f(\cos\theta)-f(\cos\theta))}{\sin^2\frac{\phi}{2}}\, d\psi_n(\phi)||_X = O(1) \qquad (n\to\infty),$$

where

$$d\psi_n(\phi) = \frac{(\alpha+\beta+2)\sin^2\frac{\phi}{2}\, d\mu_n(\cos\phi)}{(\alpha+1)(1-\overset{\vee}{\mu}_n(1))}\,.$$

By (2.9) $\int_0^\pi d\psi_n(\phi) = 1$, $n = 1,2,\ldots$ and consequently it is clear that $f \in F(X,L_n)$, if $f \in \text{Lip}(2,X)$ (see (1.14)).

We still have to prove that $f \in F(X,L_n)$ implies $f \in \text{Lip}(2,X)$. If we denote by A the operator defined by (1.14), then we will first show that for $f \in D(A)$ satisfying

$$(4.1) \qquad ||f-L_n(f)||_X \leq M(1-\overset{\vee}{\mu}_n(1)) \qquad (n\to\infty),$$

the following inequality is valid:

$$(4.2) \qquad ||Af||_X \leq C(M+||f||_X).$$

Here C is a constant independent of f.

Since the measures ψ_n all have norm 1, there exists a subsequence $\{n_j\}$ and a measure ψ such that $\{\psi_{n_j}\}$ converges weak* to ψ. By condition B and the weak* convergence it follows that for each $\varepsilon > 0$

$$(4.3) \qquad \int_0^\varepsilon d\psi = \lim_{j\to\infty} \int_0^\varepsilon d\psi_{n_j} \geq C_B.$$

We choose ε_0 so small that $\varepsilon_0 \leq \frac{\pi}{2}$ and

$$(4.4) \qquad \int_{(0,\varepsilon_0)} d\psi \leq \frac{C_B}{S} \qquad \text{with } S > 2+2^{\beta+2}.$$

For $f \in D(A)$ satisfying (4.1) we have

$$||\int_0^\pi \frac{T_\phi f - f}{\sin^2 \frac{\phi}{2}}\, d\psi(\phi)||_X \leq$$

$$\leq \lim_{j\to\infty} ||\int_0^\pi \frac{T_\phi f - f}{\sin^2 \frac{\phi}{2}}\, d\psi_{n_j}(\phi)|| \leq M.$$

Hence,

$$(4.5)\qquad \left|\left|\int_0^{\varepsilon_0} \frac{T_\phi f - f}{\sin^2\frac{\phi}{2}}\, d\psi(\phi)\right|\right|_X \leq M + \left|\left|\int_{\varepsilon_0}^{\pi} \frac{T_\phi f - f}{\sin^2\frac{\phi}{2}}\, d\psi(\phi)\right|\right|_X \leq$$

$$\leq M + \frac{2||f||_X}{\sin^2\frac{\varepsilon_0}{2}}$$

From (1.18) and (1.20) we know that $\dfrac{T_\phi f - f}{\sin^2\frac{\phi}{2}} \to -\dfrac{1}{\alpha + 1} Af$ in X if $\phi \to 0^+$.
In virtue of (4.3) and (4.4)

$$(4.6)\qquad \left|\left|\int_0^{\varepsilon_0} \frac{T_\phi f - f}{\sin^2\frac{\phi}{2}}\, d\psi(\phi)\right|\right|_X \geq$$

$$\geq (1 - \tfrac{1}{S})\, C_B \frac{1}{\alpha + 1}\, ||Af||_X - \left|\left|\int_{(0,\varepsilon)} \frac{T_\phi f - f}{\sin^2\frac{\phi}{2}}\, d\psi(\phi)\right|\right|_X.$$

Since by (1.17) and (1.21)

$$\left|\left|\frac{T_\phi f - f}{\sin^2\frac{\phi}{2}}\right|\right|_X \leq \frac{2^{\beta+1}}{\alpha + 1}\, ||Af||_X\,, \qquad 0 < \phi \leq \frac{\pi}{2}\,,$$

we derive from (4.6) and (4.4)

$$(4.7)\qquad \left|\left|\int_0^{\varepsilon_0} \frac{T_\phi f - f}{\sin^2\frac{\phi}{2}}\, d\psi(\phi)\right|\right|_X \geq$$

$$\geq (1 - \tfrac{1}{S})\, C_B \frac{1}{\alpha + 1}\, ||Af||_X - \frac{C_B}{S}\frac{2^{\beta+1}}{\alpha + 1}\, ||Af||_X \geq$$

$$\geq \frac{1}{2(\alpha+1)}\, C_B\, ||Af||_X,$$

as we have chosen $S > 2+2^{\beta+2}$.

Hence (4.7) and (4.5) yield

$$||Af||_X \leq \frac{2(\alpha+1)}{C_B} \left(M + \frac{2||f||_X}{\sin^2 \frac{\varepsilon_0}{2}}\right) ,$$

which establishes (4.2).

If we take an arbitrary element of $F(X,L_n)$ such that

$$||f-L_n(f)||_X \leq M(1-\overset{\vee}{\mu}_n(1)) \qquad (n=1,2,\ldots),$$

then we study the convolution of f with a positive polynomial kernel K_m (for instance the de la Vallée-Poussin kernel (see section (5.1)) $f_m = f * K_m$, which clearly belongs to D(A). Then for f_m

$$||f_m-L_n(f_m)||_X = ||f*K_m-f*K_m*d\mu_n||_X = ||(f-f*d\mu_n)*K_m||_X \leq$$

$$\leq ||f-f*d\mu_n||_X \leq M(1-\overset{\vee}{\mu}_n(1)) \qquad (n=1,2,\ldots).$$

Since $||f_m||_X \leq ||f||_X$ holds, it follows from (4.2) that

$$||Af_m||_X \leq C(M+||f_m||_X) \leq C(M+||f||_X).$$

Hence for $\phi > 0$ it follows from (1.17) and (1.21)

$$(4.8) \qquad ||\frac{T_\phi f_m - f_m}{\phi^2}||_X \leq \frac{2^{\beta-1}}{\alpha+1} ||Af_m||_X \leq C_1(M+||f||_X), \qquad (m=1,2,\ldots).$$

If we take the limit as $m \to \infty$ in we get

$$||\frac{T_\phi f - f}{\phi^2}||_X \leq C_1(M+||f||_X)$$

which is equivalent with $f \in Lip(2,X)$.

5. Applications

We will show in this section, that many of the classical approximation processes which have a positive kernel, satisfy the conditions of theorem 2.3. Since condition (a) of theorem 2.3 is essentially stronger than condition A of theorem 1.5, we may conclude by theorem 1.5, that these approximation processes are saturated with order $(1-\check{\mu}_n(1))$ and that their saturation class in Lip(2,X). For some of the examples given here, these results have already been obtained by different methods in Bavinck [2].

5.1. The de la Vallée-Poussin summability process

The de la Vallée-Poussin kernel is defined by

$$V_N(\cos\theta) = \omega_0^{(\alpha,\beta+N)}(\cos\frac{\theta}{2})^{2N} \qquad N = 1,2,\ldots, \tag{5.1}$$

where $\omega_0^{(\alpha,\beta+N)}$ is given in (1.3).
The trigonometric moments of V_N are very easy to calculate:

$$T(V_N;2\sigma) = \frac{\omega_0^{(\alpha,\beta+N)}}{\omega_0(\alpha+\sigma,\beta+N)} .$$

Hence

$$\lim_{N\to\infty}\frac{T(V_N;4)}{T(V_N;2)} = \lim_{N\to\infty}\frac{\omega_0^{(\alpha+1,\beta+N)}}{\omega_0^{(\alpha+2,\beta+N)}} = \lim_{N\to\infty}\frac{\alpha+2}{N+\alpha+\beta+3} = 0.$$

By theorem 2.3 and theorem 1.5 we conclude that the summability process $V_N f(\cos\theta) = (f*V_N)(\cos\theta)$ is saturated with the order $1 - \hat{V}_N(1)$, which by (2.9) is

$$1 - \hat{V}_N(1) = \frac{(\alpha+\beta+2)}{(\alpha+1)}\,T(V_N;2) = \frac{\alpha+\beta+2}{N+\alpha+\beta+2} .$$

The saturation class is Lip(2,X).

5.2 The Jackson kernel

We now direct our attention to the Jackson kernel

(5.2) $$L_{n,r}(\theta) = \lambda_{n,r}^{-1}\left(\frac{\sin n\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)^{2r} \qquad (r \text{ and } n \text{ positive integers, } r > \alpha+2),$$

where

$$\lambda_{n,r} = \int_0^\pi \left(\frac{\sin n\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)^{2r} \rho^{(\alpha,\beta)}(\theta)\, d\theta \approx n^{2r-2\alpha-2}.$$

In order to find the saturation order and the saturation class, we show that the kernel (5.2) satisfies condition B of theorem 1.4. Using the well-known estimates $\frac{\theta}{\pi} \leq \sin\frac{\theta}{2} \leq \frac{\theta}{2}$ for $0 \leq \theta \leq \pi$ and $\frac{\sqrt{2}}{2} \leq \cos\frac{\theta}{2} \leq 1$ for $0 \leq \theta \leq \frac{\pi}{2}$ we have

$$\lambda_{n,r}.T(L_{n,r};2) = \int_0^\pi \frac{(\sin n\frac{\theta}{2})^{2r}}{(\sin\frac{\theta}{2})^{2r-2\alpha-3}} (\cos\frac{\theta}{2})^{2\beta+1}\, d\theta \leq$$

$$\leq \pi^{2r-2\alpha-3} (\frac{n}{2})^{2r} \int_0^{\frac{\pi}{n}} \theta^{2\alpha+3}\, d\theta + \pi^{2r-2\alpha-3} \int_{\frac{\pi}{n}}^{\pi} \theta^{2\alpha+3-2r}\, d\theta \leq$$

$$\leq (\frac{\pi}{2})^{2r+1} \frac{n^{2r-2\alpha-4}}{\alpha+2} + \frac{n^{2r-2\alpha-4}}{2r-2\alpha-4}\,\pi =$$

$$= n^{2r-2\alpha-4} \left((\frac{\pi}{2})^{2r+1} \frac{1}{\alpha+2} + \frac{\pi}{2r-2\alpha-4}\right).$$

On the other hand ($n \geq 2$)

$$\lambda_{n,r} \int_0^{\frac{\pi}{n}} \frac{(\sin n\frac{\theta}{2})^{r}}{(\sin\frac{\theta}{2})^{2r-2\alpha-3}} (\cos\frac{\theta}{2})^{2\beta+1}\, d\theta \geq (\frac{n}{\pi})^{2r}\, 2^{2r-2\alpha-3-\beta-\frac{1}{2}} \int_0^{\frac{\pi}{n}} \theta^{2\alpha+3}\, d\theta =$$

$$= (\frac{n}{\pi})^{2r-2\alpha-4} \frac{2^{2r-2\alpha-4-\beta-\frac{1}{2}}}{(\alpha+2)}.$$

If we choose $\varepsilon > 0$, then for $n > \frac{\pi}{\varepsilon}$

$$\int_0^{\varepsilon} L_{n,r}(\theta)\, \rho^{(\alpha,\beta)}(\theta)\, d\theta \geq C_B \int_0^{\pi} L_{n,r}(\theta)\, \rho^{(\alpha,\beta)}(\theta)\, d\theta,$$

where

$$C_B = \frac{2^{2r-2\alpha-4-\beta-\frac{1}{2}}}{\pi^{2r-2\alpha-4}(\alpha+2)} \left(\left(\frac{\pi}{2}\right)^{2r+1} \frac{1}{\alpha+2} + \frac{\pi}{2r-2\alpha-4} \right)^{-1}.$$

Since $T(L_{n,r};2) \approx n^{-2}$ it follows from (2.9) and theorem 1.5 that the kernel $L_{n,r}(\theta)$ is saturated with order n^{-2} and that the saturation class is Lip(2,X).

5.3. The Weierstrass kernel

The Weierstrass kernel, defined by

$$(5.3) \qquad W_t(\cos\theta) = \sum_{k=0}^{\infty} e^{-k(k+\alpha+\beta+1)t}\, \omega_k^{(\alpha,\beta)}\, R_k^{(\alpha,\beta)}(\cos\theta) \qquad (t > 0)$$

is a positive kernel (see Bavinck [2], section 5.8). If we take a sequence of numbers $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, then it is easy to show that the sequence of convolution operators W_{t_n} satisfies condition (a) of theorem 2.3. In fact

$$\lim_{t_n \to 0^+} \frac{1 - e^{-k(k+\alpha+\beta+1)t_n}}{1 - e^{-(\alpha+\beta+2)t_n}} = \frac{k(k+\alpha+\beta+1)}{\alpha+\beta+2}.$$

Hence by theorem 1.5 the sequence W_{t_n} is saturated with order $1 - e^{-(\alpha+\beta+2)t_n} \approx t_n$ $(n\to\infty)$ and the saturation class in Lip(2,X).

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